Section A: Pure Mathematics

1 Let $x = 10^{100}$, $y = 10^x$, $z = 10^y$, and let

$$a_1 = x!, \quad a_2 = x^y, \quad a_3 = y^x, \quad a_4 = z^x, \quad a_5 = e^{xyz}, \quad a_6 = z^{1/y}, \quad a_7 = y^{z/x}.$$

(i) Use Stirling's approximation $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$, which is valid for large n, to show that $\log_{10}(\log_{10} a_1) \approx 102$.

(ii) Arrange the seven numbers $a_1, \ldots, a_7$ in ascending order of magnitude, justifying your result.

2 Consider the quadratic equation

$$nx^2 + 2x\sqrt{(pn^2 + q)} + rn + s = 0, \qquad (*)$$

where $p > 0$, $p \neq r$ and $n = 1, 2, 3, \ldots$.

(i) For the case where $p = 3$, $q = 50$, $r = 2$, $s = 15$, find the set of values of n for which equation $(*)$ has no real roots.

(ii) Prove that if $p < r$ and $4q(p - r) > s^2$, then $(*)$ has no real roots for any value of n.

(iii) If $n = 1$, $p - r = 1$ and $q = s^2/8$, show that $(*)$ has real roots if, and only if, $s \leqslant 4 - 2\sqrt{2}$ or $s \geqslant 4 + 2\sqrt{2}$.

3 Let

$$\mathrm{S}_n(x) = e^{x^3} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^3}\right).$$

Show that $\mathrm{S}_2(x) = 9x^4 - 6x$ and find $\mathrm{S}_3(x)$.

Prove by induction on n that $\mathrm{S}_n(x)$ is a polynomial. By means of your induction argument, determine the order of this polynomial and the coefficient of the highest power of x.

Show also that if $\dfrac{\mathrm{d}S_n}{\mathrm{d}x} = 0$ for some value a of x, then $S_n(a)S_{n+1}(a) \leqslant 0$.

4 By considering the expansions in powers of x of both sides of the identity

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

show that

$$\sum_{s=0}^{n} \binom{n}{s}^2 = \binom{2n}{n},$$

where $\displaystyle\binom{n}{s} = \frac{n!}{s!\,(n-s)!}$.

By considering similar identities, or otherwise, show also that:

(i) if n is an even integer, then

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s}^2 = (-1)^{n/2}\binom{n}{n/2};$$

(ii) $$\sum_{t=1}^{n} 2t\binom{n}{t}^2 = n\binom{2n}{n}.$$

5 Show that if α is a solution of the equation

$$5\cos x + 12\sin x = 7,$$

then either

$$\cos\alpha = \frac{35 - 12\sqrt{120}}{169}$$

or $\cos\alpha$ has one other value which you should find.

Prove carefully that if $\pi/2 < \alpha < \pi$, then $\alpha < 3\pi/4$.

6 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if

$$y = \frac{ax+b}{cx+d}. \qquad (*)$$

By using changes of variable of the form $(*)$, or otherwise, show that

$$\int_0^1 \frac{1}{(x+3)^2}\,\ln\left(\frac{x+1}{x+3}\right)\mathrm{d}x = \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12},$$

and evaluate the integrals

$$\int_0^1 \frac{1}{(x+3)^2}\,\ln\left(\frac{x^2+3x+2}{(x+3)^2}\right)\mathrm{d}x \quad \text{and} \quad \int_0^1 \frac{1}{(x+3)^2}\,\ln\left(\frac{x+1}{x+2}\right)\mathrm{d}x.$$

7 The curve C has equation

$$y = \frac{x}{\sqrt{(x^2 - 2x + a)}} \ ,$$

where the square root is positive. Show that, if $a > 1$, then C has exactly one stationary point.

Sketch C when (i) $a = 2$ and (ii) $a = 1$.

8 Prove that

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} \ . \qquad (*)$$

(i) Deduce that, when n is large,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) \approx \frac{2n}{\pi} \ .$$

(ii) By differentiating $(*)$ with respect to θ, or otherwise, show that, when n is large,

$$\sum_{k=0}^{n} k \sin^2\left(\frac{k\pi}{2n}\right) \approx \left(\frac{1}{4} + \frac{1}{\pi^2}\right) n^2 \ .$$

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\,\theta^2$ may be assumed.]

Section B: Mechanics

9 In the Z–universe, a star of mass M suddenly blows up, and the fragments, with various initial speeds, start to move away from the centre of mass G which may be regarded as a fixed point. In the subsequent motion the acceleration of each fragment is directed towards G. Moreover, in accordance with the laws of physics of the Z–universe, there are positive constants k_1, k_2 and R such that when a fragment is at a distance x from G, the magnitude of its acceleration is k_1x^3 if $x < R$ and is k_2x^{-4} if $x \geqslant R$. The initial speed of a fragment is denoted by u.

(i) For $x < R$, write down a differential equation for the speed v, and hence determine v in terms of u, k_1 and x for $x < R$.

(ii) Show that if $u < a$, where $2a^2 = k_1R^4$, then the fragment does not reach a distance R from G.

(iii) Show that if $u \geqslant b$, where $6b^2 = 3k_1R^4 + 4k_2/R^3$, then from the moment of the explosion the fragment is always moving away from G.

(iv) If $a < u < b$, determine in terms of k_2, b and u the maximum distance from G attained by the fragment.

10 N particles P_1, P_2, P_3, …, P_N with masses m, qm, q^2m, … , $q^{N-1}m$, respectively, are at rest at distinct points along a straight line in gravity-free space. The particle P_1 is set in motion towards P_2 with velocity V and in every subsequent impact the coefficient of restitution is e, where $0 < e < 1$. Show that after the first impact the velocities of P_1 and P_2 are

$$\left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad \left(\frac{1+e}{1+q}\right)V,$$

respectively.

Show that if $q \leqslant e$, then there are exactly $N-1$ impacts and that if $q = e$, then the total loss of kinetic energy after all impacts have occurred is equal to

$$\frac{1}{2}me\big(1-e^{N-1}\big)V^2.$$

11 An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius R in a horizontal plane at a constant angular speed ω. A shell is fired from O, the centre of this circle, with initial speed V and angle of elevation α. Show that if $V^2 < gR$, then no matter what the value of α, or what vertical plane the shell is fired in, the shell cannot hit the target.

Assume now that $V^2 > gR$ and that the shell hits the target, and let β be the angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that β satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

Deduce that there are exactly two possible values of β.

Let β_1 and β_2 be the possible values of β and let P_1 and P_2 be the corresponding points of impact. By considering the quantities $(\beta_1^2 + \beta_2^2)$ and $\beta_1^2\beta_2^2$, or otherwise, show that the linear distance between P_1 and P_2 is

$$2R\sin\left(\frac{\omega}{g}\sqrt{(V^2 - Rg)}\right) .$$

Section C: Probability and Statistics

12 It is known that there are three manufacturers A, B, C, who can produce micro chip MB666. The probability that a randomly selected MB666 is produced by A is $2p$, and the corresponding probabilities for B and C are p and $1-3p$, respectively, where $0 \leqslant p \leqslant \frac{1}{3}$. It is also known that 70% of MB666 micro chips from A are sound and that the corresponding percentages for B and C are 80% and 90%, respectively.

Find in terms of p, the conditional probability, $\mathrm{P}(A|S)$, that if a randomly selected MB666 chip is found to be sound then it came from A, and also the conditional probability, $\mathrm{P}(C|S)$, that if it is sound then it came from C.

A quality inspector took a random sample of one MB666 micro chip and found it to be sound. She then traced its place of manufacture to be A, and so estimated p by calculating the value of p that corresponds to the greatest value of $\mathrm{P}(A|S)$. A second quality inspector also a took random sample of one MB666 chip and found it to be sound. Later he traced its place of manufacture to be C and so estimated p by applying the procedure of his colleague to $\mathrm{P}(C|S)$.

Determine the values of the two estimates and comment briefly on the results obtained.

13 A stick is broken at a point, chosen at random, along its length. Find the probability that the ratio, R, of the length of the shorter piece to the length of the longer piece is less than r.

Find the probability density function for R, and calculate the mean and variance of R.

14 You play the following game. You throw a six-sided fair die repeatedly. You may choose to stop after any throw, except that you must stop if you throw a 1. Your score is the number obtained on your last throw. Determine the strategy that you should adopt in order to maximize your expected score, explaining your reasoning carefully.